1. From Textbook 3.2. To an n-bit word with a single parity bit (for a total of (n + 1) bits) a second parity bit for the (n+1)-bit word has been added. How would the error detection capabilities change?

**Key.** The first parity bit detects any odd number of errors in the (n + 1) bits, including the check bit. A second parity bit detects any odd number of errors in the (n + 2) bits of the longer codeword. The only additional errors that are detected by the second parity bit are the errors that modify this second parity bit. Another way to see is this to note that the second parity bit of every codeword is 0, so it does not increase the minimum distance of the simple parity code.

2. Consider an 8-bit data. How many parity bits are needed to design a Hamming Distance-3 code (to correct one-bit error) and how many parity bits are needed to design a Distance-5 code (to correct two-bit errors).

**Key.** The total number bits is 8+r where r is the number of parity bits

a). For detecting 1 bit error (distance 3 code), we need to represent 8+r+1 states or $2^r \geq 8+r+1$ and if we use $r=4$, this will be satisfied

b). For detecting 2 bit errors we need
   1. no error state
   2. 8+r one bit errors
   3. $(8+r)^n(8+r-1)/2$ 2-bit errors

Total number states = 1 + 8+r + $(8+r)^n(8+r-1)/2$
If we try $r=7$, the number of states needed = 121 and we can encode these using $r=7$ parity bits

3. For the problem #2, find the probability that there are more than one bit error when using distance-3 code and the probability that there are more than two bit errors when using distance-5 code. These probabilities can be viewed as the failure probabilities

**Key:**
For distance 3 code. Assume we have a total of n bits and the probability that one bit is in error is $p$.

The probability of more than 1 error = 1 – (no errors) – (one bit error)

$= 1 - (1-p)^n - n*p*(1-p)^{n-1}$

In the previous problem, $n=12$
Probability of more than 1 bit in error
For distance 5 code. Probability of more than 2 errors
\[ = 1 - (1-p)^{12} - 12p(1-p)^{11} \]

From previous problem, \( n = 15 \)
Probability of more than 2 errors
\[ = 1 - (1-p)^{15} - 15p^2(1-p)^{14} - 105p^2(2-p)^{13} \]

4. Consider the binary field \( \text{GF}(2) \). That is the set of \{0,1\} with modulo 2 arithmetic. Consider an \( n \) dimensional vector space over this field, that is \( V(n,2) \).

Now consider a subspace that consists of only those vectors of even weight. That is, it consists of only \( n \) bit numbers where the number of 1's is even.

Is this a linear subspace? That is, is the vector addition and scalar multiplication closed?

Write a basis for the subspace. If you wish, you can use a 4-dimensional space (ie, 4-bit binary numbers) as examples.

What is the minimum weight of vectors in this space (or minimum distance between any pair of vectors)?

**Key.** Note the scalar multiplication does not cause much of a problem since the only two multipliers are 0 and 1.

Consider adding two binary numbers each with even number of 1's. The position of 1's in the two numbers may match (and adding these numbers removes the 1's with a zero under modulo 2 arithmetic) or they are in different positions (in which case the number of 1's in the sum will increase by an even number).

For example, take two 4 bit binary vectors: 0110 and 1001
The result of modulo 2 addition will be 1111. Or if you add 0110 to 1100, the result will be 1010.

Note we are not looking the numbers as binary numbers but as binary vectors.

So, the binary vectors with even number of 1's forms a subspace.

The basis will consist of 0000...0 and other vectors chosen such that the even number of ones in the basis vectors do not overlap in their bit positions. For example consider the 4-bit binary numbers. We can choose 0000, 1010 and 0101 as the basis (not with 4 bits, the maximum number of codewords with even number of ones is 8).
b). Note that there must be a codeword (other than zero) whose weight is the minimum distance. Since in our vector space with even number of ones, we can consider a codeword with 2 ones. Thus the minimum distance is 2.